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Goldstone bosons in Josephson junctions

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Abstract
For a Josephson junction model where the φ^4 interaction is seen as a pseudo-spin interaction, a non-equilibrium steady state is rigorously constructed showing the occurrence of a Josephson current. The main new contribution in this paper is the rigorous construction of two dynamic Goldstone bosons which arise due to the symmetry breaking. The normal coordinates of the two-junction bosons are constructed and their dynamical spectrum is computed. The explicit dependence on the phase difference of the two superconductors is calculated.

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1. Introduction

In 1962, Josephson [1] predicted a novel phenomenon in superconductivity, namely when two different superconductors are brought into close contact. Based on elementary quantum mechanics, he predicted the existence of a supercurrent with a peculiar current–voltage dependence. He argued that a current of Cooper pairs would emerge which is proportional to the sine of the phase difference of the order parameters of both superconductors. The success of this prediction was immediate when indeed this phenomenon was experimentally observed one year later [2]. It counts as one of the greatest successes of quantum mechanics in physics and one will find a chapter on the Josephson effects in almost every textbook on superconductivity. The increase of knowledge on this subject in theoretical solid state physics in the following decades was tremendous and applications of Josephson junctions in electronic devices were developed [3]. Progress in conceiving a microscopic theory for the Josephson effect in rigorous quantum statistical mechanics was made when Sewell obtained the Josephson and Meissner effects in a model-independent approach from the assumption of off-diagonal long-range order and local gauge covariance [4]. Further rigorous work about the Josephson junction coupling Hamiltonian has been done, e.g., by Rieckers and co-workers [14, 15] using the algebraic approach as we also use below. As far as the physics of these models based on field theoretic considerations is concerned, we refer to [16].

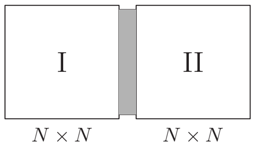

Figure 1. Two superconductors wit a contact surface.

In [5] we considered a concrete quantum model yielding a rigorous understanding of the emergence of a Josephson current, which is computed and numerically calculated. The model consists of two two-dimensional superconducting plates having a common one-dimensional contact surface through which Cooper pairs can tunnel and as such induce a current (figure 1). Cooper pairs are the well-known bounded pairs of electrons with opposite spin and momentum forming the basis of the BCS theory of superconductivity. In section 2 we repeat the essentials of the model. We construct a non-equilibrium steady state (NESS). One of the attractive aspects of the construction is that our NESS has the nice property of having a finite interaction area. We derive an analytical expression for the current in the case that the phases of the two superconductors are not too large. We find back the perfect sine behaviour.

Our main contribution is contained in section 3 which is devoted to the study of the appearance of Goldstone bosons in the junction, due to the interaction of the two bulk superconductors and as a direct consequence of the gauge symmetry breaking. We apply the general result of [7] where one finds the general explicit construction of the normal coordinates of the Goldstone boson as a consequence of the spontaneous symmetry breaking which appears in our model. The Goldstone bosons of the bulk superconductors can also be found in that paper. Here we find two supplementary Goldstone bosons. We construct their normal coordinates and their dynamics induced by the micro-dynamics of the model. We consider their dynamics in the diagonal form and analyse its spectrum, again as in section 2, as a function of the phase difference of the two bulk superconductors. Here one finds a cosine behaviour.

2. Model for Josephson junctions

As said above we consider the model [5] for two two-dimensional superconducting plates I and II with a common one-dimensional contact surface (line) through which the Cooper pairs can travel in order to induce a current (figure 1).

The two superconductors are modelled by the strong coupling BCS model on a square lattice using the Anderson quasi-spin formalism (see [16]), described by the Hamiltonians $H_{i,N}$ with $i = I, II$

$$H_{i,N} = \sum_{k,l=1}^{N} \epsilon_i \sigma^z(k,l) - \frac{1}{N} \sum_{k,l,m,n=1}^{N} \sigma_i^+(k,l)\sigma_i^-(m,n), \qquad \epsilon_i > 0 \tag{2.1}$$

acting on the Hilbert space $\otimes_{j=1}^{N^2} \mathbb{C}_j^2$; $\sigma_i^{\pm}$ and $\sigma^z = \sigma_i^+\sigma_i^- - \sigma_i^-\sigma_i^+$ are copies of the Pauli matrices. σ^+ and σ^- represent the creation and annihilation operators of the Cooper pairs of the superconductors; ϵ represent the kinetic energies of the Cooper pairs.

The junction between the superconductors I and II is modelled by the interaction

$$V_N = -\frac{\gamma}{N} \sum_{k_1,k_2=1}^{N} \left(\sigma_I^+(k_1,1)\sigma_{II}^-(k_2,1) + \text{h.c.}\right), \qquad \gamma > 0 \tag{2.2}$$

which is responsible for the Cooper pair tunnelling through the barrier. A pair at the site $(k_I, 1)$ of the first superconductor can tunnel through the junction and create a pair at the site $(k_{II}, 1)$ of the second superconductor and vice versa. The coupling constant γ governs the rate of this process. Note that only Cooper pairs on the contact surfaces of I and II participate in this process. Remark that only N sites of each superconductor are interacting with N sites of the other one. The lattice permutation invariance of the Hamiltonians (2.1) and the interaction (2.2) makes the model, given by the total Hamiltonian of the system

$$H_N = H_{I,N} + H_{II,N} + V_N, \tag{2.3}$$

exactly soluble in the thermodynamic limit N tending to infinity [6].

2.1. Equilibrium states of the non-interacting superconductors

For completeness we discuss here the equilibrium states of the Hamiltonians (2.1). We treat the first (I) one explicitly; the second is analogous and obtained by replacing the index I by the index II. Following [6], the extremal equilibrium states at inverse temperature β_I in the thermodynamic limit are the product states ω_{φ_I} with the expectation values of all tensor product observables $X = X_{x_1} \otimes X_{x_2} \otimes \cdots, x_1, x_2, \ldots \in \mathbb{N}^2$ and all $X_{x_j} \in M_2$ (2 by 2 complex matrices), given by

$$\omega_{\varphi_I}(X) = \prod_{x \in \mathbb{N}^2} \operatorname{Tr} \rho_{\varphi_{I,x}} X_x. \tag{2.4}$$

Here $\rho_{\varphi_{I,x}}$ is the x-copy of the 2×2 density matrix $\rho_{\varphi_I} \in M_2$, solution of the self-consistency equation

$$\rho_{\varphi_I} = \frac{\exp -\beta_I h_{\varphi_I}}{\operatorname{Tr} \exp -\beta_I h_{\varphi_I}}, \tag{2.5}$$

with the one-site effective Hamiltonian h_{φ_I} given by

$$h_{\varphi_I} = \epsilon_I \sigma_I^z - \lambda_I (\mathrm{e}^{\mathrm{i}\varphi_I} \sigma_I^- + \mathrm{h.c.}), \qquad \lambda_I \geqslant 0. \tag{2.6}$$

Clearly the density matrix (2.5) is also determined by the equivalent self-consistency equation for the order parameter $\lambda_I = \left|\omega_{\varphi_I}(\sigma_I^-)\right|$, which by explicit computation becomes

$$\lambda_I \left(1 - \frac{1}{\mu_I} \tanh \beta_I k_I\right) = 0, \qquad \mu_I = \sqrt{\epsilon_I^2 + \lambda_I^2}. \tag{2.7}$$

We remark that $\{\pm\mu_I\}$ constitutes the spectrum of the effective Hamiltonian h_{φ_I} which is independent of the phase angle φ_I.

It can readily be seen that (2.1) admits always a solution $\lambda_I = 0$. It yields the *normal phase state* of the superconductor. For $\varepsilon_I < \frac{1}{2}$ and β_I being large enough or temperature low enough, there exists also a solution $\lambda_I \neq 0$. These solutions yield the superconducting phase states. The phase φ_I can be fixed freely, yielding an infinite degeneration of the equilibrium states under the conditions mentioned above.

The second superconductor(II) has analogous phase states with phases which can be chosen independently from the first one.

In the following, we fix for each superconductor such a superconducting phase state denoted by ω_{I,φ_I} and $\omega_{II,\varphi_{II}}$ with $\varphi_I \neq \varphi_{II}$.

2.2. Non-equilibrium steady state (NESS)

Now we construct a non-equilibrium but steady state (NESS) for the total interacting system (2.3). We start from the product state $\omega = \omega_{I,\varphi_I} \otimes \omega_{II,\varphi_{II}}$ on the system. Each of the states

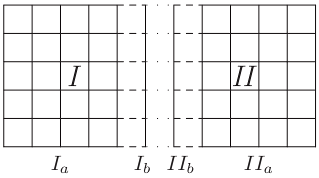

Figure 2. Division of the system in four subsystems.

of the system of the two superconductors in their respective superconducting phase states are characterized by their phases, inverse temperatures and kinetic energies. In this state, ω, one can compute the global dynamics yielding a time evolution

$$\alpha_t(.) = \omega - \lim_N (\exp \mathrm{i}t H_N \exp \mathrm{i}t H_N), \tag{2.8}$$

where $\omega - \lim$ is the weak operator limit under the state ω for N going to infinity. Now we are ready to look for a state $\tilde{\omega}$ which is invariant under this dynamics.

Due to the specific lattice permutation symmetry of the model, it is natural to choose this state $\tilde{\omega}$ among the product states [6], i.e.

$$\tilde{\omega}_{\varphi_I}(X) = \mathrm{Tr} \prod_{x\in\mathbb{N}^2} \tilde{\rho}_x X_x. \tag{2.9}$$

As the state should be time invariant, it should have the same lattice invariance as the Hamiltonian (2.3). The symmetry of the Hamiltonian divides the total system into four parts, see figure 2.

We denote the bulk parts of the superconductors I and II by I_a and II_a, and the contact or surface parts by I_b and II_b. Therefore one can write the state $\tilde{\omega}$ as a tensor product of four symmetric product states on their different regions:

$$\tilde{\omega} = \tilde{\omega}_{I_a} \otimes \tilde{\omega}_{I_b} \otimes \tilde{\omega}_{II_b} \otimes \tilde{\omega}_{I_a}. \tag{2.10}$$

Finally we require that the state $\tilde{\omega}$ is a steady state

$$\lim_N \tilde{\omega}([H_N, X]) = 0. \tag{2.11}$$

Due to the product structure of the state, the Hamiltonian can be identified with an effective Hamiltonian [6] of the type $\tilde{H}_N = \sum_{x\in I_N\cup II_N} \tilde{h}_x$, where $\tilde{h}_x \in M_{2,x}$. Imposing this time invariance yields

$$\tilde{h}_i = \begin{cases} \epsilon_I \sigma_I^z(i) - \tilde{\omega}\big(\sigma^+_{I_a}\big)\sigma^-_{I_a}(i) - \tilde{\omega}\big(\sigma^-_{I_a}\big)\sigma^+_{I_a}(i), & i \in I_a; \\ \epsilon_I \sigma^z_{I_b}(i) - \tilde{\omega}\big(\sigma^+_{I_a}\big)\sigma^-_{I_b}(i) - \tilde{\omega}\big(\sigma^-_{I_a}\big)\sigma^+_{I_b}(i) & \\ \quad - \gamma\big(\tilde{\omega}\big(\sigma^+_{II_b}\big)\sigma^-_{I_b}(i) + \tilde{\omega}\big(\sigma^-_{II_b}\big)\sigma^+_{I_b}(i)\big), & i \in I_b; \\ \epsilon_{II} \sigma^z_{II_b}(i) - \tilde{\omega}\big(\sigma^+_{II_a}\big)\sigma^-_{II_b}(i) - \tilde{\omega}\big(\sigma^-_{II_a}\big)\sigma^+_{II_b}(i) & \\ \quad - \gamma\big(\tilde{\omega}\big(\sigma^+_{I_b}\big)\sigma^-_{II_b}(i) + \tilde{\omega}\big(\sigma^-_{I_b}\big)\sigma^+_{II_b}(i)\big), & i \in II_b; \\ \epsilon_{II} \sigma^z_{II_b}(i) - \tilde{\omega}\big(\sigma^+_{II_a}\big)\sigma^-_{II_b}(i) - \tilde{\omega}\big(\sigma^-_{II_a}\big)\sigma^+_{II_b}(i), & i \in II_a. \end{cases} \tag{2.12}$$

We also use the notation

$$\tilde{\Lambda}_I = \tilde{\lambda}_I \exp i\tilde{\varphi}_I = \tilde{\omega}\big(\sigma^+_{I_b}\big), \qquad \tilde{\varphi}_I = \arg \tilde{\omega}\big(\sigma^+_{I_b}\big) \tag{2.13}$$

and analogously for the second superconductor with I replaced by II.

The local density matrices $\tilde{\rho}_x$ for $x \in I \cup II$ of the state $\tilde{\omega}$ are the $\tilde{h}_x$ invariant projections of the density matrices ρ_x of ω (2.4). For more details of the construction, see [5]. In any case, for $x \in I_a \cup II_a : \rho_x = \tilde{\rho}_x$ as follows from (2.12). For the lattice point $x \in I_b \cup II_b$ one readily computes the self-consistency non-equilibrium equations

$$\begin{aligned}\tilde{\lambda}_I\,\mathrm{e}^{\mathrm{i}\tilde{\phi}_I} &= (\lambda_I\,\mathrm{e}^{\mathrm{i}\phi_I} + \gamma\tilde{\lambda}_{II}\,\mathrm{e}^{\mathrm{i}\tilde{\phi}_{II}})\frac{\epsilon_I^2 + \lambda_I|\lambda_I\,\mathrm{e}^{\mathrm{i}\phi_I} + \gamma\tilde{\lambda}_{II}\,\mathrm{e}^{\mathrm{i}\tilde{\phi}_{II}}|\cos(\phi_I - \tilde{\phi}_I)}{\epsilon_I^2 + |\lambda_I\,\mathrm{e}^{\phi_I} + \gamma\tilde{\lambda}_{II}\,\mathrm{e}^{\mathrm{i}\tilde{\phi}_{II}}|^2}\\ \tilde{\lambda}_{II}\mathrm{e}^{\mathrm{i}\tilde{\phi}_{II}} &= (\lambda_{II}\,\mathrm{e}^{\mathrm{i}\phi_{II}} + \gamma\tilde{\lambda}_I\,\mathrm{e}^{\mathrm{i}\tilde{\phi}_I})\frac{\epsilon_{II}^2 + \lambda_{II}|\lambda_{II}\,\mathrm{e}^{\mathrm{i}\phi_{II}} + \gamma\tilde{\lambda}_I\,\mathrm{e}^{\mathrm{i}\tilde{\phi}_I}|\cos(\phi_{II} - \tilde{\phi}_{II})}{\epsilon_{II}^2 + |\lambda_{II}\,\mathrm{e}^{\phi_{II}} + \gamma\tilde{\lambda}_I\,\mathrm{e}^{\mathrm{i}\tilde{\phi}_I}|^2}.\end{aligned} \tag{2.14}$$

Together with the self-consistency equations (2.1) for λ_I and λ_{II}, equations (2.14) form a set of six coupled equations whose solutions determine the non-equilibrium steady state $\tilde{\omega}$ of the total system.

This state divides the system into four parts. The bulk parts of both superconductors away from the contact surface do not feel each other nor do they feel the surface. They behave as stable reservoirs. On the contact surfaces I_b and II_b the system is effectively perturbed and influenced by the properties of the states of both superconductors.

We remark that we limited the interaction to take place only on a contact surface of one layer thickness. It is clear that the whole construction can be generalized to the case of any fixed finite number of layers.

In [5] we considered the currents of Cooper pairs emerging in the system tunnelling from the superconductor I to II and vice versa. One considers the relative particle number operator of the Cooper pairs

$$Q_N = \sum_{x\in N^2}\left(\sigma_I^+\sigma_I^- - \sigma_{II}^+\sigma_{II}^-\right). \tag{2.15}$$

The local relative current is then

$$J(Q_N) = i[H_N, Q_N] = -\frac{2\mathrm{i}\gamma}{N}\sum_{i,j}^{N}\left(\sigma_{I_b}^-(i,1)\sigma_{II_b}^+(j,1) - \text{h.c.}\right). \tag{2.16}$$

We remark that in the observable current there is no direct contribution from the bulk of the two superconductors, only the two contact layers are contributing. Moreover, one remarks that this current is of the same order of magnitude as the contact surface, namely N.

The Josephson current measured in the thermodynamic limit ($N \to \infty$) state $\tilde{\omega}$, called NESS, is readily calculated and given by

$$j(Q) = \lim_N \frac{\tilde{\omega}(J(Q_N))}{N} = -4\gamma\tilde{\lambda}_I\tilde{\lambda}_{II}\sin(\tilde{\varphi}_I - \tilde{\varphi}_{II}). \tag{2.17}$$

This result was obtained in [5]; here it is written in a more concise form. However, one has to realize that the quantities $\tilde{\lambda}$ and $\tilde{\varphi}$ are functions of the originally given parameters $\lambda_I, \lambda_{II}, \varphi_I$ and φ_{II}, given by (2.14). First of all it is easy to see that these equations are shift invariant for an arbitrary shift of the two originally given angles. This means that, without loss of generality, one can take one of the angels, say φ_I, equal to zero. Furthermore, it is natural to assume the coupling constant γ to be very small, i.e. $\gamma \ll \min(\epsilon_I, \epsilon_{II})$. Therefore it is reasonable to compute the quantities $\tilde{\lambda}$ and $\tilde{\varphi}$ only up to first order in this parameter γ.

By multiplying the two self-consistency equations with each other, using the fact that $\varphi_I = 0$ and taking $\varphi_{II} < \frac{\pi}{2}$, one gets that the phase difference $\tilde{\varphi}_I - \tilde{\varphi}_{II}$ is proportional to the phase difference $\varphi_I - \varphi_{II}$. Suppose now that also the second phase φ_{II} is small, corresponding to the usual experimental regime. Then one also remarks that $\tilde{\varphi}_I = 0$. It follows that

$$\tilde{\varphi}_{II} \approx \varphi_{II}. \tag{2.18}$$

After substitution of (2.18) in (2.14) one gets

$$\tilde{\lambda}_I = \lambda_I - \gamma \frac{\lambda_I^2 \lambda_{II}}{\mu_I^2} \qquad \tilde{\lambda}_{II} = \lambda_{II} + \gamma \frac{\lambda_I^2 \epsilon_{II}^2}{\mu_{II}^2}. \tag{2.19}$$

After substitution of all these equations in formula (2.17) one gets the expected formula for the Josephson current

$$j(Q) = -4\gamma \lambda_I \lambda_{II} \sin(\varphi_I - \varphi_{II}) \tag{2.20}$$

yielding an analytical expression for the current for small phase differences between the two bulk superconductors. A numerical computation of the current for arbitrary phase differences is found in [5].

3. Symmetry breaking and Goldstone bosons

As is well known, spontaneous symmetry breakdown (SSB) is one of the basic features accompanying collective phenomena. It is a representative tool for the analysis of many phenomena in modern physics. For long range interactions, SSB is typically accompanied also by the breaking of the symmetry of the dynamics. The latter phenomenon is known to be accompanied by the occurrence of oscillations of a Goldstone boson with a non-vanishing energy spectrum. These oscillations together with the Goldstone boson disappear if the SSB disappears. In [7] one was able to construct explicitly the normal coordinates of these new particles called Goldstone particles. In particular for mean field systems such as the BCS model [8], the Overhauser model [9], a spin density wave model [10], the anharmonic crystal model [13] and the jellium model [11], one has constructed these Goldstone boson normal coordinates. It should be remarked that in the case of long-range interactions (e.g. mean fields case) the Goldstone boson is often called a plasmon.

Our two-dimensional model consisting of two interacting superconductors also shows the phenomenon of SSB. As the main contribution of this paper we consider the construction and the calculation of the spectrum of the corresponding Goldstone bosons.

As far as the Josephson current, computed in the previous section, is concerned, we remark that the bulk parts I_a and II_a of the superconductors are not contributing to it. Therefore it is reasonable to look for the Goldstone particles within the contact areas I_b and II_b. In particular we compute the normal coordinates and the dynamics of the Goldstone bosons of this junction area. From [7] we know that the canonical coordinates of these bosons are given by the fluctuation operators of the generator of the broken symmetry and of the order parameter operator.

As the following gauge transformation holds

$$\mathrm{e}^{\mathrm{i}\alpha\sigma^z} \sigma^+ \mathrm{e}^{-\mathrm{i}\alpha\sigma^z} = \sigma^+ \mathrm{e}^{2\mathrm{i}\alpha}, \qquad \alpha \in \mathbb{R}, \tag{3.1}$$

σ^z are the local generators of the broken gauge symmetry of the effective Hamiltonian (2.12). Indeed for all α

$$\tilde{\omega}(\mathrm{e}^{\mathrm{i}\alpha\sigma^z} \sigma^+ \mathrm{e}^{-\mathrm{i}\alpha\sigma^z}) = \tilde{\omega}(\sigma^+)\, \mathrm{e}^{2\mathrm{i}\alpha} \tag{3.2}$$

proving that the state is not invariant under the $U(1)$ gauge group.

Therefore we consider the local operators, for $i \in I_b, II_b$

$$\tilde{Q}_i = \frac{|\tilde{\Lambda}_i^2|}{\tilde{\mu}_i^2} \sigma_i^z + \frac{\epsilon_i}{\tilde{\mu}_i^2} \left(\overline{\tilde{\Lambda}}_i \sigma_i^+ + \mathrm{h.c.} \right), \tag{3.3}$$

where for all $i \in I_b$:

$$\tilde{\Lambda}_i = \tilde{\Lambda}_{I_b} = \tilde{\omega}(\sigma^+_{I_b}) + \gamma\tilde{\omega}(\sigma_{II_b}) \tag{3.4}$$

$$\tilde{\mu}_i = \tilde{\mu}_{I_b} = \sqrt{\epsilon_I{}^2 + |\tilde{\Lambda}_{I_b}|^2} \tag{3.5}$$

$$\epsilon_I = \epsilon_i \tag{3.6}$$

and equivalently with $i \in II_b$, i.e. by substitution of I by II and vice versa.

We remark that the operator $\tilde{Q}_i$ is indeed the generator of the gauge transformations, namely up to a constant equal to σ^z, but normalized to zero expectation value:

$$\tilde{\omega}(\tilde{Q}_i) = \frac{|\tilde{\Lambda}_i{}^2|}{\tilde{\mu}_i{}^2}\tilde{\omega}(\sigma^z) + \frac{\epsilon_i}{\tilde{\mu}_i{}^2}2|\tilde{\Lambda}_i|^2 = 0. \tag{3.7}$$

We also consider essentially the order parameter operator $\sigma^{\pm}$ fluctuation

$$\tilde{P}_j = \frac{i}{\tilde{\mu}_j}\left(\overline{\tilde{\Lambda}_j}\sigma^+_j - \text{h.c.}\right). \tag{3.8}$$

Again we remark that $\tilde{\omega}(\tilde{P}_j) = 0$, i.e. this operator is also duly normalized.

Using the general quantum fluctuation theory for product states [12], one computes the following quantum central limits in the given state $\tilde{\omega}$ and obtain the normal coordinates of two Goldstone bosons. For the region I_b one gets the normal coordinates

$$b_{I_b}(Q) = \lim_N \frac{1}{\sqrt{N}} \sum_{j\in I_b, j=1}^{N} \tilde{Q}_j \qquad b_{I_b}(P) = \lim_N \frac{1}{\sqrt{N}} \sum_{j\in I_b, j=1}^{N} \tilde{P}_j, \tag{3.9}$$

and for the region II_b one gets the normal coordinates

$$b_{II_b}(Q) = \lim_N \frac{1}{\sqrt{N}} \sum_{j\in II_b, j=1}^{N} \tilde{Q}_j \qquad b_{II_b}(P) = \lim_N \frac{1}{\sqrt{N}} \sum_{j\in II_b, j=1}^{N} \tilde{P}_j. \tag{3.10}$$

In (3.9) one gets the normal coordinates of a first Goldstone boson, and in (3.10) the normal coordinates of a second independent Goldstone boson. Indeed, by a straightforward computation one checks readily the following canonical commutation relations:

$$\begin{gathered}[b_{I_b}(Q), b_{II_b}(Q)] = [b_{I_b}(Q), b_{II_b}(P)] = [b_{I_b}(P), b_{II_b}(Q)] = [b_{I_b}(P), b_{II_b}(P)] = 0 \\ [b_{I_b}(Q), b_{I_b}(P)] = 4\,\mathrm{i}\frac{\tilde{\lambda}^2_{I_b}}{\tilde{\mu}_{I_b}} \qquad [b_{II_b}(Q), b_{II_b}(P)] = 4\,\mathrm{i}\frac{\tilde{\lambda}^2_{II_b}}{\tilde{\mu}_{II_b}}.\end{gathered} \tag{3.11}$$

We remark that in the case of temperatures above the critical ones of the bulk superconductors the order parameters vanish, $\tilde{\lambda}_{I_b} = \tilde{\lambda}_{II_b} = 0$, such that all commutators in (3.11) vanish. Also one has

$$\tilde{\omega}(b_{I_b}(Q)^2) = \tilde{\omega}(b_{II_b}(Q)^2) = \tilde{\omega}(b_{I_b}(P)^2) = \tilde{\omega}(b_{II_b}(P)^2) = 0, \tag{3.12}$$

and hence all the operators themselves vanish, $b_{I_b}(Q) = b_{I_b}(P) = b_{II_b}(Q) = b_{II_b}(Q) = 0$, i.e. the Goldstone bosons disappear in the normal phases.

Next we consider the dynamics of the Goldstone bosons in the case of superconducting phases for the bulk superconductors. We consider the time evolution of the normal modes (3.3) and (3.8) which is induced by the initial micro-dynamics given by the effective Hamiltonian (2.12). In general, let A be a local observable situated at the lattice point $x \in I_b$ or II_b, then denote by $\tilde{\alpha}_t$ the time evolution of the fluctuation of A after time t. It is given by

$$\begin{aligned}\tilde{\alpha}_t b_{I_b}(A) &= b_{I_b}(\exp(\mathrm{i}t\tilde{h}_x)A\exp(-\mathrm{i}t\tilde{h}_x)) \\ \tilde{\alpha}_t b_{II_b}(A) &= b_{II_b}(\exp(\mathrm{i}t\tilde{h}_x)A\exp(-\mathrm{i}t\tilde{h}_x)).\end{aligned} \tag{3.13}$$

Of course the operator A stands for the operators $\tilde{Q}_x$ (3.3) and $\tilde{P}_x$ (3.8).

A straightforward computation of the dynamics using (2.8) yields the simple solutions

$$\begin{aligned}\tilde{\alpha}_t b_{I_b}(Q) &= b_{I_b}(Q)\cos\left(2\tilde{\mu}_{I_b}t\right) + b_{I_b}(P)\sin\left(2\tilde{\mu}_{I_b}t\right)\\ \tilde{\alpha}_t b_{I_b}(P) &= -b_{I_b}(Q)\sin\left(2\tilde{\mu}_{I_b}t\right) + b_{I_b}(P)\cos\left(2\tilde{\mu}_{I_b}t\right),\end{aligned} \tag{3.14}$$

and analogously for the surface II_b one gets the same dynamics for the second Goldstone boson by replacing the index I_b by the index II_b.

The two bosons behave dynamically as two independent quantum harmonic oscillators with frequencies

$$\tilde{\nu}_I = 2\tilde{\mu}_{I_b} \qquad \tilde{\nu}_{II} = 2\tilde{\mu}_{II_b}. \tag{3.15}$$

For the bulk superconductors I_a and II_a the Goldstone bosons dynamics were computed before [8]. The frequencies $\nu_I = 2\mu_I$ and $\nu_{II} = 2\mu_{II}$ are clearly phase independent.

However, for the frequencies of the Goldstone bosons considered in this paper, the situation is completely different. The frequencies (3.15) computed above depend on the phase difference $\varphi_I - \varphi_{II}$ of the two phases of the bulk superconductors.

We consider the phase dependence explicitly for the region I_b, the computation for the second region is analogous:

$$\tilde{\nu}_I = 2\sqrt{\epsilon_I^2 + \left|\tilde{\Lambda}_{I_b}\right|^2} = 2\sqrt{\epsilon_I^2 + |\lambda_I \exp \mathrm{i}\varphi_I + \gamma\tilde{\lambda}_{II}\exp \mathrm{i}\tilde{\varphi}_{II}|^2}. \tag{3.16}$$

We remark that $\tilde{\lambda}_I$ and $\tilde{\varphi}_{II}$ are determined by the parameters $\epsilon_I, \epsilon_{II}, \lambda_I, \lambda_{II}$ and the phases φ_I and φ_{II} through the self-consistency equations (2.14).

It is instructive again to get an explicit form of the frequencies in terms of the given parameters of the system. We derive again the formula for the frequency $\tilde{\nu_I}$ up to the first order in the coupling constant γ and in the case that the phase difference $\varphi_I - \varphi_{II}$ is small, a situation explored before for the current.

We get from (2.18), (2.19) and (3.15)

$$\tilde{\mu}_I^2 = \epsilon_I^2 + \lambda_I^2 + 2\gamma\lambda_I\lambda_{II}\cos(\varphi_I - \varphi_{II}). \tag{3.17}$$

Hence one gets the following expressions for the dynamical frequencies of the Goldstone modes:

$$\tilde{\nu_I} = \nu_I + 4\gamma\frac{\lambda_I\lambda_{II}}{\nu_I}\cos(\varphi_I - \varphi_{II}) \tag{3.18}$$

$$\tilde{\nu_{II}} = \nu_{II} + 4\gamma\frac{\lambda_I\lambda_{II}}{\nu_{II}}\cos(\varphi_I - \varphi_{II}). \tag{3.19}$$

From these expressions the dependence of the Goldstone frequencies on the phase differences of the bulk superconductors is explicitly given. We learn that these frequencies decrease when the phase difference increases in contradistinction with the current. The current has a sine behaviour, the frequencies a cosine behaviour. This point may be interesting from the experimental point of view.

References

[1] Josephson B D 1962 Possible new effects in superconductors *Phys. Lett.* **1** 251–3

[2] Anderson P W and Rowell J M 1963 Probable observation of the Josephson superconducting tunneling effect *Phys. Rev. Lett.* **10** 230–2

[3] Likharev K K 1986 *Dynamics of Josephson Junctions and Circuits* (New York: Gordan and Breach)

[4] Sewell G L 1997 Off-diagonal long range order and superconductive electrodynamics *J. Math. Phys.* **38** 2053–71

[5] Lauwers J and Verbeure A 2004 A microscopic model for Josephson currents *J. Phys. A: Math. Gen.* **37** 3577–87

[6] Fannes M, Spohn H and Verbeure A 1980 Equilibrium states for mean-field models *J. Math. Phys.* **21** 355–8
[7] Michoel T and Verbeure A 2001 Goldstone boson normal coordinates *Commun. Math. Phys.* **216** 461–90
[8] Goderis D, Verbeure A and Vets P 2004 Fluctuation oscillations and symmetry breaking *Il Nuovo Cimento* **106** 375–82
[9] Broidioi M, Nachtergaele B and Verbeure A 1991 The Overhauser model *J. Math. Phys.* **32** 2929–35
[10] Broidioi M and Verbeure A 1991 Plasmon frequency for a spin density wave model *Helv. Phys. Acta* **64** 1093–112
[11] Broidioi M and Verbeure A 1993 The plasmon of the one-component plasma *Helv. Phys. Acta* **66** 155–80
[12] Goderis D, Verbeure A and Vets P 1989 Non-commutative central limits *Prob. Theor. Relat. Fields* **82** 527–44
[13] Verbeure A and Zagrebnov V A 1992 Phase-transitions and algebra of fluctuation operators in an exactly soluble model of a quantum anharmonic crystal *J. Stat. Phys.* **69** 329–59
[14] Rieckers A and Ullrich M 1986 On the microscopic derivation of the finite temperature Josephson relation in operator form *J. Math. Phys.* **27** 1082–92
[15] Unnerstall T and Rieckers A 1989 Quasispin operator despription of the Josephson tunnel junction and the Josephson plasma frequency *Phys. Rev.* B **39** 2173–9
[16] Lee P A and Scully M O 1971 Theory of Josephson radiation I. General theory *Phys. Rev.* B **3** 769–79